VEKTORSKI PROSTORI

$V \neq \emptyset$

F, polje

$+ : V^2 \to V$ unutrašnja operacija nad V

$\cdot : F \times V \to V$ preslikavanje (vanjska)

Skup V je vektorski prostor nad poljem F ako:

1) $(V, +)$ Abelova grupa

2) $\alpha \cdot (x+y) = \alpha \cdot x + \alpha \cdot y$

$(\alpha+\beta) \cdot x = \alpha \cdot x + \beta \cdot x$

$\alpha(\beta \cdot x) = (\alpha \cdot \beta) \cdot x$

$1 \cdot x = x$

$\forall \alpha, \beta \in F$, $\forall x, y \in V$, 1 je jedinični element u F

24. Neka je $S \neq \emptyset$ i F polje skalara. Dokazati da je $F^S \overset{def}{=} \{f : S \to F\}$ vektorski prostor nad poljem F za $(\forall f, g \in F^S)$ $f+g$ je def. sa: $(f+g)(x) = f(x) +_F g(x)$, $\forall x \in S$, $(\forall a \in F)(\forall f \in F^S)$ $a \cdot f$ je def. sa: $(a \cdot f)(x) = a \cdot_F f(x)$, $\forall x \in S$.

$(F^S, +_F, \cdot_F)$

1) $(F^S, +)$ Abelova grupa

a) zatvorenost: $(\forall f, g \in F^S)$ $f+g \in F^S$; $f+g : S \to F$ ✓

b) asocijativnost: $(\forall f, g, h \in F^S)$ $f+(g+h) = (f+g)+h$

(dvije fje su jednake ako za svaki element domena imaju istu vrijednost)

$x \in S;\ (f+(g+h))(x) = f(x) +_F (g+h)(x) = f(x) +_F (g(x) +_F h(x))$

$= (f(x) +_F g(x)) +_F h(x) = (f+g)(x) +_F h(x) = ((f+g)+h)(x)$

$+_F$ je asoc. oper. (polje skalara)

c) neutralni element: ? $(\exists e \in F^S)(\forall f \in F^S) \quad e+f = f+e = f$

rješavamo jedne okrugu treba provjeriti

$(\forall x \in S)\ (f+e)(x) = f(x)$

$f(x) +_F e(x) = f(x)$ možemo poništiti jer smo u polju skalara

$e(x) = 0 \quad \checkmark \quad (0_F)$

d) inverzni element: $(\forall f \in F^S)(\exists f' \in F^S)$

$f + f' = f' + f = e$

$(\forall x \in S)\ (f'+f)(x) = e(x)$

$f'(x) +_F f(x) = 0_F$

$f'(x) = -f(x) \quad \checkmark$

e) komutativnost: $(\forall f, g \in F^S) \quad f+g = g+f$

$(\forall x \in S)\ (f+g)(x) = f(x) +_F g(x) = g(x) +_F f(x) = (g+f)(x)$ ✓

$\Rightarrow f+g = g+f$

2) $(\forall \alpha, \beta \in F)(\forall f, g \in F^S)$

a) $\alpha \cdot (f+g) = \alpha \cdot f + \alpha \cdot g$

$x \in S \quad (\alpha \cdot (f+g))(x) = \alpha \cdot_F (f+g)(x) = \alpha \cdot_F (f(x) +_F g(x))$ $\cdot_F$ distribu. prema $+_F$

$= (\alpha \cdot_F f(x)) + (\alpha \cdot_F g(x)) = (\alpha \cdot f)(x) + (\alpha \cdot g)(x) =$

$= (\alpha f + \alpha g)(x)$ ✓

b) $(\alpha +_F \beta) \cdot f = \alpha \cdot f + \beta \cdot f$

$x \in S \quad ((\alpha +_F \beta) \cdot f)(x) = (\alpha +_F \beta) \cdot_F f(x) = (\alpha \cdot_F f(x)) +_F (\beta \cdot_F f(x)) =$

$= (\alpha \cdot f)(x) +_F (\beta \cdot f)(x) = (\alpha f + \beta f)(x)$ ✓

c) $((\alpha \cdot_F \beta) \cdot f) = \alpha \cdot (\beta \cdot f)$

$(\forall x \in S)$ $\underline{((\alpha \cdot_F \beta) \cdot f)(x)} = (\alpha \cdot_F \beta) \cdot_F f(x) \overset{\cdot_F \text{ je asocijativno}}{=} \alpha \cdot_F (\beta \cdot_F f(x)) = \alpha \cdot_F (\beta \cdot f)(x) =$

$= \underline{(\alpha \cdot (\beta \cdot f))(x)}$ ✓

d) $1 \cdot f = f$ (1 je jedinični el. u polju F)

$(\forall x \in S)$ $\underline{(1 \cdot f)(x)} = 1 \cdot_F f(x) = \underline{f(x)}$ ✓

$\Rightarrow F^S$ je v.p. nad poljem F!

25. Dokazati da je skup $\mathbb{R}^+$ (svih pozitivnih realnih br.) vektorski prostor nad poljem $\mathbb{R}$, ako su operacije sabiranje vektora i množenje vektora sa skalarom definisane sa:

$(\forall u, v \in \mathbb{R}^+)(\forall \alpha \in \mathbb{R})$ $(\mathbb{R}, +, \cdot)$

$u \oplus v = u \cdot v$

$\alpha \odot v = v^\alpha$

1) $(\mathbb{R}^+, \oplus)$ je Abelova grupa

a) zatvorenost $(\forall u, v \in \mathbb{R}^+)$ $\underbrace{u \oplus v}_{u \cdot v \in \mathbb{R}^+} \in \mathbb{R}^+$

b) asoc. $(\forall u, v, w \in \mathbb{R}^+)$ $u \oplus (v \oplus w) = (u \oplus v) \oplus w$

$u \oplus (v \oplus w) = u \oplus (v \cdot w) = u \cdot (v \cdot w) \overset{\cdot \text{ je asoc.}}{=} (u \cdot v) \cdot w = (u \oplus v) \cdot w =$

$= (u \oplus v) \oplus w$

c) neutralni el. ? $(\exists e \in \mathbb{R}^+)(\forall u \in \mathbb{R}^+)$ $e \oplus v = \underline{v \oplus e = v}$

$v \cdot e = v \Rightarrow e = 1$ ✓

d) inverzni el. ? $(\forall u \in \mathbb{R}^+)(\exists u' \in \mathbb{R}^+)$ $u \oplus u' = \underline{u' \oplus u = 1}$

$u' \cdot u = 1 \underset{u>0}{} \quad u' = \frac{1}{u} \in \mathbb{R}^+$

e) komutativnost $(\forall u, v \in \mathbb{R}^+)$ $u \oplus v = v \oplus u$

$u \oplus v = u \cdot v \overset{\cdot \text{ je komut.}}{=} v \cdot u = v \oplus u$

2) $(\forall \alpha, \beta \in \mathbb{R})(\forall u, v \in \mathbb{R}^+)$

a) $\alpha \odot (u \oplus v) \overset{?}{=} \alpha \odot u \oplus \alpha \odot v$

$$\alpha \odot (u \oplus v) = \alpha \odot (u \cdot v) = (u \cdot v)^\alpha = u^\alpha \cdot v^\alpha = u^\alpha \oplus v^\alpha$$

$$= (\alpha \odot u) \oplus (\alpha \odot v)$$

b) $(\alpha + \beta) \odot v = (\alpha \odot v) \oplus (\beta \odot v)$

$$(\alpha + \beta) \odot v = v^{\alpha+\beta} = v^\alpha \cdot v^\beta = v^\alpha \oplus v^\beta = (\alpha \odot v) \oplus (\beta \odot v)$$

c) $(\alpha \cdot \beta) \odot v = \alpha \odot (\beta \odot v)$

$$(\alpha \cdot \beta) \odot v = v^{\alpha \cdot \beta} = (v^\beta)^\alpha = (\beta \odot v)^\alpha = \alpha \odot (\beta \odot v)$$

d) $1 \odot u = u$ 1 je realan br.

$$u^1 = u \quad \text{w}$$

$\Rightarrow \mathbb{R}^+$ je v.p. nad $\mathbb{R}$

ZADAĆA:

26. Neka je $F^m = \{(a_1, \dots, a_m) \mid a_i \in F\}$ skup svih uređenih m-torki skalara (F je polje). U skupu F^m definišimo operacije sabiranja vektora i množenje vektora sa skalarom sa:

$$(a_1, a_2, \dots a_m) + (b_1, \dots b_m) = (a_1 + b_1, \dots, a_m + b_m)$$

$$\alpha \cdot (a_1, \dots, a_m) = (\alpha \cdot a_1, \dots \alpha \cdot a_m)$$

Dokazati da je F^m vektorski prostor nad poljem F.

27. Neka je $V = \mathbb{R}^2$ ispitati da li je V vektorski prostor nad poljem $\mathbb{R}$ ako su definisane operacije sabiranja vektora i množenje vektora skalarom sa: $(a, b) + (c, d) = (a + c, b + d)$

$$\alpha \cdot (a, b) = (\alpha^2 \cdot a, \alpha^2 \cdot b)$$

za $\forall (a, b), (c, d) \in V$, $\forall \alpha \in \mathbb{R}$

1) $(V, +)$ Abelova grupa

a) ✓

b) $(\forall (a,b),(c,d),(e,f) \in V)\ (a,b)+((c,d)+(e,f)) = ((a,b)+(c,d))+(e,f)$

$(a,b)+((c,d)+(e,f)) = (a,b)+(c+e, d+f) = (a+(c+e), b+(d+f))$

na osnovu asoc. + u polju $\mathbb{R}$:

$= ((a+c)+e, (b+d)+f) = (a+c, b+d) + (e,f)$

$= ((a,b)+(c,d))+(e,f)$

c) .... $e = (0,0)$

d) .... $(a,b)' = (-a,-b)$

e) ✓ važi zbog komut. u polju $\mathbb{R}$

2) $(\forall \alpha, \beta \in \mathbb{R})\ (\forall (a,b),(c,d) \in V)$

a) $\underline{\alpha \cdot ((a,b)+(c,d))} = \alpha \cdot (a+c, b+d) = (\alpha^2 \cdot (a+c), \alpha^2 \cdot (b+d)) =$

$= (\alpha^2 a + \alpha^2 c, \alpha^2 b + \alpha^2 d) = (\alpha^2 a, \alpha^2 b) + (\alpha^2 c, \alpha^2 d) = \underline{\alpha(a,b) + \alpha \cdot (c,d)}$

b) $(\alpha \cdot \beta) \cdot (a,b) = \alpha \cdot (\beta \cdot (a,b))$

$((\alpha \cdot \beta)^2 \cdot a, (\alpha \cdot \beta)^2 \cdot b) = (\alpha^2 \cdot \beta^2 \cdot a, \alpha^2 \beta^2 \cdot b) =$

$= \alpha \cdot (\beta^2 \cdot a, \beta^2 \cdot b) = \alpha(\beta \cdot (a,b))$

c) $(\alpha + \beta) \cdot (a,b) = \alpha(a,b) + \beta(a,b)$

$((\alpha+\beta)^2 \cdot a, (\alpha+\beta)^2 b) = (\alpha^2 a, \alpha^2 b) + (\beta^2 a, \beta^2 b)$

— // — $= (\alpha^2 a + \beta^2 a, \alpha^2 b + \beta^2 b)$

— // — $\neq ((\alpha^2 + \beta^2) a, (\alpha^2 + \beta^2) b)$

Npr. $\alpha = \beta = 1$ $(4a, 4b) \neq (2a, 2b)$

d) $1 \cdot (a,b) \stackrel{?}{=} (a,b)$

$(1^2 a, 1^2 b) = (a,b)$

$(a,b) = (a,b)$

V nije vekt. prostor

28. Dokazati da aksioma $1\cdot x = x$ nije posljedica ostalih aksioma vektorskog prostora. (nije višak!)

Pr. Neka je $V \neq \emptyset$, F -> polje, $(V,+)$ Abelova grupa - Ako $\cdot : F \times V \to V$ definišemo na sljedeći način:

$$\alpha \cdot x = 0, \quad \forall \alpha \in F, \forall x \in V$$

2.1. $(\alpha+\beta)\cdot x = \alpha x + \beta x$

$$0 = 0 + 0$$

$$0 = 0 \checkmark$$

2.2. $\alpha(x+y) = \alpha x + \alpha y$

$$0 = 0 + 0$$

$$0 = 0 \checkmark$$

2.3. $\alpha\cdot(\beta\cdot x) = (\alpha\cdot\beta)\cdot x$

$$\alpha \cdot 0 = 0$$

$$0 = 0$$

2.4. $1\cdot x = x$

$0 = x, \ \forall x \in V!$ -> nije tačno

treba da važi za $\forall x \in V$

$1\cdot x = x$ -> može da važi a ne mora (nije višak)

29. V je vekt. prostor nad F; $U \subset V$, $U \neq \emptyset$

T: Za svaka $(\forall x, y \in U)$ $\forall \alpha, \beta \in F)$ $\alpha x + \beta y \in U \iff U$ je vektorski podprostor od V

29. Ispitati da li su sljedeći ~~podskupovi~~ ~~prostori~~ podprostori prostora $\mathbb{R}^{\mathbb{R}}$

a) $V_1 = \{f \in \mathbb{R}^{\mathbb{R}} \mid f(0) = 0\}$

b) $V_2 = \{f \in \mathbb{R}^{\mathbb{R}} \mid f(0) = 1\}$

$(\alpha f + \beta g)(0) = \alpha f(0) + \beta g(0) = \alpha \cdot 1 + \beta \cdot 1$

~~$F^S = \{f : S \to F\}$ 24. zad.~~

~~$\mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}\}$~~

a) $f, g \in V_1$; $\alpha, \beta \in \mathbb{R}$

$(\alpha f + \beta g)(0) = (\alpha f)(0) + (\beta g)(0) = \alpha \cdot f(0) + \beta \cdot g(0) = \alpha \cdot 0 + \beta \cdot 0 = 0$

$\alpha f + \beta g \in V_1$ ✓ $\Rightarrow$ V_1 je vekt. prostor

b) $\forall f, g \in V_2$, $\forall \alpha, \beta \in \mathbb{R}$

$(\alpha f + \beta g)(0) = \ldots = \alpha \cdot f(0) + \beta \cdot g(0) = \alpha \cdot 1 + \beta \cdot 1 = \alpha + \beta = 1$

$\forall \alpha, \beta \in \mathbb{R}$!?

$\alpha = \beta = 1$ $\perp$

V_2 nije v. prostor

30. a) Dokazati da je $\mathbb{R}^{[a,b]} = \{f : [a,b] \to \mathbb{R}\}$ vektorski prostor nad poljem $\mathbb{R}$ ako je $f + g$ definisan za $\forall x \in [a,b]$ sa $(f+g)(x) = f(x) + g(x)$, a $\alpha \cdot f$ $\forall x \in [a,b]$ $(\alpha \cdot f)(x) = \alpha \cdot f(x)$ za $\forall f, g \in \mathbb{R}^{\mathbb{R}}$ i $\forall \alpha \in \mathbb{R}$

b) Dokazati da $C[a,b]$ vektorski pp. nad poljem $\mathbb{R}$
skup neprekidnih f-ja

a) F^S, $F = \mathbb{R}$, $S = [a,b]$ ... vektorski prostor (24. zad.)

b) $C[a,b] \subset \mathbb{R}^{[a,b]}$;

Uzmimo bilo koje $f, g \in C[a,b]$ i $\forall \alpha, \beta \in \mathbb{R}$

$\alpha f + \beta g \in C[a,b]$ zbir 2 nepr. f-je na int. je nepr. f-ja

$C[a,b]$ v. podprostor prostora $\mathbb{R}^{[a,b]} \Rightarrow \mathbb{R}^{[a,b]}$ prostor

31. a) Neka je $M_m^n(\mathbb{R})$ skup svih matrica formata $m \times n$ čiji su elementi realni brojevi.
Ako sabiranje matrica i množenje matrica realnim br. definišemo na uobičajeni način, dokazati da je $M_m^n(\mathbb{R})$ vektorski prostor.

b) Neka je $M_m^s(\mathbb{R})$ skup svih realnih simetričnih matrica reda n. Dokazati da je $M_m^s(\mathbb{R})$ vektorski prostor. (sabiranje i množenje matrica realnim br. definisano na uobičajeni način

a) D.Z. Skup svih matrica nad bilo kojim poljem je vektorski prostor.

b) $M_m^n(\mathbb{R})$ je vekt. prostor (dio onog pod a))

$M_m^s(\mathbb{R}) \subset M_m^n(\mathbb{R})$; $(\forall A,B \in M_m^s(\mathbb{R}))(\forall \alpha,\beta \in \mathbb{R})$

$\underbrace{\alpha \cdot A + \beta \cdot B}_{=C} \in M_m^s$? $(a_{ij} = a_{ji})$

$c_{ij} = \alpha \cdot \underset{a_{ji}}{a_{ij}} + \beta \cdot \underset{b_{ji}}{b_{ij}} = \alpha a_{ji} + \beta b_{ji} = c_{ji} \Rightarrow C = \alpha A + \beta B$ sim $\in M_m^s(\mathbb{R})$

$\Rightarrow M_m^s(\mathbb{R})$ v. podprostor prostora $M_m^n(\mathbb{R})$

32. Ispitati lin. zavisnost vektora $x=(1,2,3)$, $y=(2,5,7)$, $z=(3,7,1)$
$x, y, z \in \mathbb{R}^3$.

$a_1, a_2, \ldots, a_m \in V$; $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0 \Rightarrow \alpha_1 = \ldots = \alpha_m = 0$
$\alpha_1, \ldots, \alpha_m \in F$; $a_1, \ldots, a_m$ lin. nez.

$\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$ $\quad \alpha_i \neq 0$ lin. zav.

$\alpha_1 x + \alpha_2 y + \alpha_3 z = 0$

$\alpha_1 \cdot (1,2,3) + \alpha_2 (2,5,7) + \alpha_3 (3,7,1) = (0,0,0)$

$$\left.\begin{aligned} \alpha_1 + 2\alpha_2 + 3\alpha_3 &= 0 \\ 2\alpha_1 + 5\alpha_2 + 7\alpha_3 &= 0 \\ 3\alpha_1 + 7\alpha_2 + \alpha_3 &= 0 \end{aligned}\right\} \text{homogeni sis.}$$

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 1 \end{vmatrix} = -9 \neq 0 \rightarrow \text{sis. ima samo trivijalno rj.}$$

tj. $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$

33. Naći dimenziju i bar jednu bazu vektorskog prostora
$\mathbb{R}^m = \{(a_1, a_2, \ldots, a_m) \mid a_1, \ldots, a_m \in \mathbb{R}\}$.

V vekt. p.; $B \subset V$; $B \neq \emptyset$; ~~baza~~ B je baza vektorskog prostora V ako: 1) B je lin. nez. skup
2) $L(B) = V$

$L(\{x_1, x_2, \ldots, x_m\}) = \{\sum_{i=1}^{m} \alpha_i x_i \mid \alpha_i \in F\}$

$S \subset V$; $L(S) = \{\sum_{i=1}^{m} \alpha_i x_i \mid x_i \in S, \alpha_i \in F, m \in \mathbb{N}\}$

$L(S) = V$

$\dim V = |B|$, B konačna

$B = \{(1,0,\ldots,0), (0,1,\ldots,0), \ldots (0,\ldots,0,1)\}$

1) B lin. nez. ?

$\alpha_1 \cdot e_1 + \alpha_2 \cdot e_2 + \ldots + \alpha_m e_m = 0$

$\vdots$

$(\alpha_1, \alpha_2, \ldots, \alpha_m) = (0, 0, \ldots, 0) \Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_m = 0 \Rightarrow e_1, \ldots e_m$ lin. nez.

2) $L(B) = \mathbb{R}^m$?

$(a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$

$(a_1, a_2, \ldots, a_m) = a_1 \cdot e_1 + a_2 \cdot e_2 + \ldots + a_m e_m \subset L(B)$

$L(B) \subset \mathbb{R}^m$ - podrazumijeva se

$\Rightarrow L(B) = \mathbb{R}^m$

$\Rightarrow \dim \mathbb{R}^m = m$

34. Naći dim. i bar jednu bazu vektorskog prostora $M_m^m(\mathbb{R})$

$$E_{11} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$E_{12} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$E_{ij} = \left[\begin{array}{c|c} 0 & 0 \\ \hline & 1 \\ 0 & 0 \end{array}\right]$$

$\vdots$

$$E_{mm} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

1) $\alpha_{11} E_{11} + \ldots + \alpha_{mm} E_{mm} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix} \Rightarrow \ldots \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & & \\ \alpha_{m1} & \cdots & \alpha_{mm} \end{bmatrix} = [0]_{m \times m}$

$\Rightarrow \alpha_{ij} = 0$ za koje $\forall i, j$

tj. $E_{11}, \ldots E_{mm}$ lin. nez.

2) $A = [a_{ij}]_{m\times n} \in M_m^m(\mathbb{R})$

$\Rightarrow A = \ldots = \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} E_{ij} \in L(B)$

$\Rightarrow B = \{E_{11}, \ldots, E_{mm}\}$ je baza

$\dim M_m^m(\mathbb{R}) = m \cdot m$

(35) $M_m^S(\mathbb{R})$ bar jednu bazu

$E_{11} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ $E_{12} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & & 0 \\ \vdots & & & & \\ 0 & \cdots & & & 0 \end{bmatrix}$ $E_{1m} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$

$E_{22} = \begin{bmatrix} 0 & 0 & \\ 0 & 1 & \\ & & \end{bmatrix}$ ... $E_{mm} = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}$

$\dim M_m^S(\mathbb{R}) = m + (m-1) + (m-2) + \ldots + 1 = \frac{m\cdot(m+1)}{2}$

(provesti kroz 2 zahtjeva da li je baza)

36. Da li je $B = \{(1,0,0), (1,1,0), (1,1,1)\}$ baza vekt. p. $\mathbb{R}^3$?
Ako jeste odrediti koordinate vektora $v = (1,7,2)$
u odnosu na bazu.

$(1,0,0) = v_1$, $(1,1,0) = v_2$, $(1,1,1) = v_3$

- B baza?

$\alpha_1(1,0,0) + \alpha_2(1,1,0) + \alpha_3(1,1,1) = (0,0,0)$

$D \neq 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \Rightarrow$ lin. nezavisni v_1, v_2, v_3

$\{v_1, v_2, v_3\}$ generiše $\mathbb{R}^3$?

$(a_1, a_2, a_3) \in \mathbb{R}^3 \quad (a_1, a_2, a_3) = \alpha_1 \cdot v_1 + \alpha_2 v_2 + \alpha_3 v_3 =$

$= \ldots = (\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_3)$

$$\left.\begin{aligned} \alpha_1+\alpha_2+\alpha_3&=a_1\\ \alpha_2+\alpha_3&=a_2\\ \alpha_3&=a_3\end{aligned}\right\} \qquad \begin{aligned}\alpha_1&=a_1-a_2\\ \alpha_2&=a_2-a_3\\ \alpha&=a_3\end{aligned}$$

$\Rightarrow (a_1,a_2,a_3)=(a_1-a_2)\cdot v_1+(a_2-a_3)v_2+a_3\cdot v_3\in L(B)$

$\Rightarrow B$ je baza $\mathbb{R}^3$

$(1,7,2)=\alpha_1\cdot v_1+\alpha_2 v_2+\alpha_3 v_3=\ldots=(\alpha_1+\alpha_2+\alpha_3,\ \alpha_2+\alpha_3,\ \alpha_3)$

$$\left.\begin{aligned} \alpha_1+\alpha_2+\alpha_3&=1\\ \alpha_2+\alpha_3&=7\\ \alpha_3&=2\end{aligned}\right\} \qquad \begin{aligned}\alpha_1&=-6\\ \alpha_2&=5\\ \alpha_3&=2\end{aligned} \quad \Rightarrow [v]_B=(-6,5,2)$$

37. Neka je V realan vektorski prostor svih simetričnih matrica reda 2. Odrediti koordinate vektora $M=\begin{bmatrix}7&-9\\-9&6\end{bmatrix}$ u odnosu na bazu $B=\left\{\underset{A_1}{\begin{bmatrix}1&-1\\-1&2\end{bmatrix}},\underset{A_2}{\begin{bmatrix}3&-2\\-2&1\end{bmatrix}},\underset{A_3}{\begin{bmatrix}-1&3\\3&-3\end{bmatrix}}\right\}$

Prethodno provjeriti da li je to zaista baza ovog vekt. prostora.

1) $\alpha_1\cdot A_1+\alpha_2\cdot A_2+\alpha_3\cdot A_3=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

D.Z. (n. jed. - 2 iste)

$\alpha_1=\alpha_2=\alpha_3=0 \Rightarrow \{A_1,A_2,A_3\}$ lin. nez.

2) $\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}=\alpha_1\cdot A_1+\alpha_2\cdot A_2+\alpha_3\cdot A_3 \ldots$ D.Z. $\alpha_1,\alpha_2,\alpha_3$

$\in L(B)$

$$\begin{bmatrix} 7 & -9 \\ -9 & 6 \end{bmatrix} = \alpha_1 \cdot A_1 + \alpha_2 \cdot A_2 + \alpha_3 \cdot A_3$$

$$\left.\begin{aligned} \alpha_1 + 3\alpha_2 - \alpha_3 &= 7 \\ -\alpha_1 - 2\alpha_2 + 3\alpha_3 &= -9 \\ -\alpha_1 - 2\alpha_2 + 3\alpha_3 &= -9 \\ 2\alpha_1 + 4\alpha_2 - 3\alpha_3 &= 6 \end{aligned}\right\} \ldots \quad \alpha_1 = -15,\ \alpha_2 = 6,\ \alpha_3 = -4$$

$$\Rightarrow [M]_B = (-15, 6, -4)$$

$$B' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \qquad [M]_{B'} = (7, -9, 6)$$

Ako se neki vektor v može napisati:

$v = \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m$

$B = \{a_1, a_2, \ldots, a_m\}$

onda koord. v u odnosu na bazu $\rightarrow [v]_B = (\alpha_1, \ldots, \alpha_m)$

38) Neka je V vekt. prostor polinoma stepena ≤ 3

Ispitati lin. zavisnost vektora:

a) $t^3 - 4t^2 + 2t + 3,\ t^3 + 2t^2 + 4t - 1,\ 2t^3 - t^2 - 3t + 5$

b) $\underbrace{t^3 - 5t^2 - 2t + 3}_{u},\ \underbrace{t^3 - 4t^2 - 3t + 4}_{v},\ \underbrace{2t^3 - 7t^2 - 7t + 9}_{w}$

$\alpha u + \beta v + \gamma w = 0$

$$\underbrace{(\alpha + \beta + \gamma)}_{=0} t^3 + \underbrace{(-5\alpha - 4\beta - 7\gamma)}_{=0} t^2 + \underbrace{(-2\alpha - 3\beta - 7\gamma)}_{=0} t + \underbrace{(3\alpha + 4\beta + 9\gamma)}_{=0} = 0$$

$(\gamma, -3\gamma, \gamma)$

lin. zav.

39. Neka su vektori x, y, z lin. nez. Ispitati lin. zavisnost vektora:

a) $x,\ x+y,\ x+y+z$

b) $x+y,\ y+z,\ z+x$ lin. nez.

c) $x-y,\ y-z,\ z-x$

a) $\alpha_1 x + \alpha_2(x+y) + \alpha_3(x+y+z) = 0$

$(\alpha_1+\alpha_2+\alpha_3)x + (\alpha_2+\alpha_3)y + \alpha_3 z = 0$

$\xRightarrow{x,y,z \text{ nez}}$ $\alpha_1+\alpha_2+\alpha_3 = 0$

$\alpha_2+\alpha_3 = 0$

$\alpha_3 = 0$

$\alpha_1 = \alpha_2 = \alpha_3 = 0 \Rightarrow x,\ x+y,\ x+y+z$ lin. nez.

c) $\alpha_1(x-y) + \alpha_2(y-z) + \alpha_3(z-x) = 0$

$$\left.\begin{aligned} \alpha_1 \quad\quad - \alpha_3 &= 0 \\ -\alpha_1 + \alpha_2 \quad\quad &= 0 \\ -\alpha_2 + \alpha_3 &= 0 \end{aligned}\right\} \Leftrightarrow \alpha_1 = \alpha_2 = \alpha_3$$

ima netrivijalnih rj.

$\Rightarrow x-y,\ y-z,\ z-x$ lin. zavisni

40. Ako su vektori $x_1, x_2, \dots x_m$ lin. nez., da li su vektori: $y_1 = x_1$

$y_2 = x_1 + x_2$

$\vdots$

$y_m = x_1 + x_2 + \dots + x_m$

lin. nezavisni?

$$\alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_m y_m = 0$$

$$(\alpha_1 + \alpha_2 + \dots + \alpha_m)x_1 + (\alpha_2 + \dots + \alpha_m)x_2 + \dots + (\alpha_{m-1} + \alpha_m)x_{m-1} + \alpha_m x_m = 0$$

$x_1, \dots, x_m$ lin. nez. $\Rightarrow$

$$\begin{aligned} \alpha_1 + \alpha_2 + \dots + \alpha_m &= 0 \\ \alpha_2 + \dots + \alpha_m &= 0 \\ &\vdots \\ \alpha_{m-1} + \alpha_m &= 0 \\ \alpha_m &= 0 \end{aligned}$$

$$D = \begin{vmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ 0 & 0 & 1 \dots & 1 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 \cdot \dots \cdot 1 = 1 \neq 0$$ $\Rightarrow$ sis. ima samo trivijalno rj.

$$\Rightarrow \alpha_1 = \dots = \alpha_m = 0$$

$$\Rightarrow y_1, \dots, y_m \text{ – lin. nez.}$$

41. U vektorskom prostoru $P_2[x]$ polinoma stepena $\deg \leq 2$ su dati vektori:

$v_1 = \alpha + x + x^2$

$v_2 = 1 + \alpha x + x^2$

$v_3 = 1 + x + \alpha x^2$

$$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} \begin{matrix} \\ \\ \cdot(-1) \quad \cdot(-\alpha) \end{matrix} = \begin{vmatrix} 0 & 1-\alpha & 1-\alpha^2 \\ 0 & \alpha-1 & 1-\alpha \\ 1 & 1 & \alpha \end{vmatrix} = (1-\alpha)^2 + (1-\alpha)(1-\alpha^2)$$

$$= (1-\alpha)^2 + (1-\alpha)(1-\alpha)(1+\alpha)$$

$$= (1-\alpha)^2(\underbrace{1+1+\alpha}_{2+\alpha}) \neq 0$$

$1-\alpha \neq 0 \Rightarrow \alpha \neq 1$

$2+\alpha \neq 0 \Rightarrow \alpha \neq -2$

Odrediti sve vrijednosti skalara α za koje su ovi vektori lin. nezavisni.

$$\sum_{i=1}^{3} \alpha_i v_i = \dots = \alpha_1\alpha + \alpha_2 + \alpha_3 + (\alpha_1 + \alpha_2\alpha + \alpha_3)x + (\alpha_1 + \alpha_2 + \alpha_3\alpha)x^2 = 0$$

izjednačimo sa nula vektorom; da bi bio polinom 0 svi c_i su $= 0$;

$$\left.\begin{aligned} \alpha_1\alpha + \alpha_2 + \alpha_3 &= 0 \\ \alpha_1 + \alpha_2\alpha + \alpha_3 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3\alpha &= 0 \end{aligned}\right\} \quad D = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = \dots = (\alpha-1)^2(\alpha+2) \neq 0$$

$$\alpha \in \mathbb{R} \setminus \{1, -2\}$$

42. Za koje vrijednosti $a \in \mathbb{R}$ vektori:
$(a, 1-a, a)$, $(2a, 2a-1, a+2)$, $(-2a, a, -a)$ čine bazu vektorskog prostora $\mathbb{R}^3$? provjeriti lin. nez.
moramo ispitati pomoću teorema:

T: ~~n-linearno nezavisnih~~ skup od n vektora n-dimenzionalnog vektorskog prostora V čine bazu vekt. p. V akko su lin. nezavisne.

$\dim \mathbb{R}^3 = 3$

$a \in \mathbb{R} \setminus \{0, 1, -\frac{4}{3}\}$

43. Naći koordinate polinoma $f(x) = a_0 + a_1 x + \dots + a_n x^n$ u odnosu na bazu $B_1 = \left\{ 1, \frac{x-c}{1!}, \frac{(x-c)^2}{2!}, \dots \frac{(x-c)^n}{n!} \right\}$

$f \in P_n[x]$ $\quad B = \{1, x, x^2, \dots x^n\}$ $\quad P_n[x] = n+1$

$$[f]_B = (a_0, a_1, \dots, a_n)$$

$f(x) = \alpha_0 + \alpha_1 \frac{x-c}{1!} + \dots + \alpha_n \frac{(x-c)^n}{n!}$ $\quad$ umjesto x stavimo povoljan br.

$f(c) = \alpha_0$

$f'(x) = \alpha_1 + \alpha_2 (x-c) + \dots + \frac{\alpha_n (x-c)^{n-1}}{(n-1)!}$ $\quad \Rightarrow f'(c) = \alpha_1$

$f''(x) = \alpha_2 + \alpha_3 (x-c) + \dots + \frac{\alpha_n (x-c)^{n-2}}{(n-2)!}$ $\quad \Rightarrow f''(c) = \alpha_2$

$\vdots$

$f^{(n)}(x) = \alpha_n$ $\quad \Rightarrow f^{(n)}(c) = \alpha_n$

Koordinate vektora f u odnosu na B_1 su:
$[f]_{B} = (f(c), f'(c), \dots, f^{(n)}(c))$

44. Odrediti bazu i dim. vektorskog prostora:

$V = L(\{a_1, a_2, a_3, a_4, a_5\})$ gdje su: $a_1 = (1,0,0,-1)$,

$a_2 = (2,1,1,0)$, $a_3 = (1,1,1,1)$, $a_4 = (1,2,3,4)$, $a_5 = (0,1,2,3)$

Formiramo mat. čije će vrste biti ovi vektori:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{matrix} /\cdot(-2) \; /\cdot(-1) \\ \\ \\ \\ \\ \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 3 & 5 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{matrix} \\ /\cdot(-1) \; /\cdot(-2) \\ \\ \\ \\ \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ /\cdot(-1) \\ \\ \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

izbacujemo nula vektore i 1 kolonu => minor

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0 \qquad \text{rang } A = 3$$

$\Rightarrow \dim V = 3$ $\qquad B = \{\underbrace{(1,0,0,-1), (0,1,1,2), (0,0,1,1)}_{\text{3 nenula vektora}}\}$

45. Naći bazu i dim. vektorskog prostora svih rješenja jednačine $x_1+x_2+\ldots+x_m=0$ u polju $\mathbb{R}$ po nepoznatim $x_1,\ldots,x_m$.

$R_0=\{(x_1,x_2,\ldots,x_m)\in\mathbb{R}^m \mid x_1+x_2+\ldots+x_m=0\}$ skup rješenja

$=\{(x_1,x_2,\ldots,x_{m-1},-x_1-x_2-\ldots-x_{m-1})\}$

$=\{x_1(1,0,\ldots,-1)+x_2(0,1,0,\ldots,0,-1)+\ldots+x_{m-1}(0,0,\ldots,0,1,-1) \mid x_1,\ldots,x_{m-1}\in\mathbb{R}\}$

$=L(\{(1,0,\ldots,0,-1),\ldots,(0,\ldots,0,1,-1)\})$

$\alpha_1(1,0,\ldots,0,-1)+\ldots+\alpha_{m-1}(0,\ldots,0,1,-1)=0$

$$\left.\begin{array}{l}\alpha_1=0\\ \alpha_2=0\\ \vdots\\ \alpha_{m-2}=0\\ \alpha_{m-1}=0\\ -\alpha_1-\alpha_2-\ldots-\alpha_{m-1}=0\end{array}\right\} \Rightarrow \alpha_1=\alpha_2=\ldots=\alpha_{m-1}=0 \Rightarrow \text{lin. nez.}$$

$\leftarrow$ kada m-tu koordinatu izjednačimo sa nula

$\Rightarrow \dim R_0=m-1$

$B_{R_0}=\{(1,0,\ldots,0,-1),\ldots,(0,\ldots,0,1,-1)\}$

46. Pokazati da je u v. prostoru $\mathbb{R}^4$ skup $S=\{(x_1,x_2,x_3,x_4) \mid x_1-3x_2-x_3=x_4=0\}$ podprostor i naći njegovu dimenziju.

$\forall x,y\in S,\ \forall\alpha,\beta\in\mathbb{R}$

$\alpha x+\beta y=\alpha(3x_2+x_3,x_2,x_3,0)+\beta(3y_2+y_3,y_2,y_3,0)$

$=\ldots=(3(\alpha x_2+\beta y_2)+(\alpha x_3+\beta y_3),\alpha x_2+\beta y_2,\alpha x_3+\beta y_3,0)\in S$

$\Rightarrow S$ je podprostor prostora $\mathbb{R}^4$

Dimenzija:

$S=\{(3x_2+x_3, x_2, x_3, 0) \mid x_2, x_3 \in \mathbb{R}\}$

$=\{x_2\cdot(3,1,0,0)+x_3(1,0,1,0) \mid x_2, x_3 \in \mathbb{R}\}$

$= L(\{(3,1,0,0),(1,0,1,0)\})$

, očigledno je da su lin. nez.

$\Rightarrow \dim S = 2$

47. U v. prostoru $\mathbb{R}^{\mathbb{R}}$ $(=\{f: \mathbb{R}\to\mathbb{R}\})$ nad poljem $\mathbb{R}$ ispitati lin. zavisnost vektora:

a) $1, x, x^2, x^3$

b) $2, \sin x, \cos x$

c) $1, \sin^2 x, \cos^2 x$

a) $\alpha_1\cdot 1+\alpha_2\cdot x+\alpha_3\cdot x^2+\alpha_4\cdot x^3=0$ (nula je f-ja)

$\alpha_1+\alpha_2 x+\alpha_3 x^2+\alpha_4 x^3=0 \quad \forall x\in\mathbb{R}$

pol. nekog stepena da bi bio $=0$ za $\forall x\in\mathbb{R}$, koeficijenti su $=0$:

$\Rightarrow \alpha_1=\alpha_2=\alpha_3=\alpha_4=0$

$\Rightarrow 1, x, x^2, x^3$ su lin. nez.

b) $\alpha_1\cdot 2+\alpha_2\sin x+\alpha_3\cos x=0, \quad \forall x\in\mathbb{R}$

$$\left.\begin{array}{ll} x=0 & 2\alpha_1+\alpha_3=0 \\ x=\frac{\pi}{2} & 2\alpha_1+\alpha_2=0 \\ x=\pi & 2\alpha_1-\alpha_3=0 \end{array}\right\} \quad D=\begin{vmatrix} 2 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{vmatrix} = -2-2=-4\neq 0$$

$\Rightarrow$ sis. ima samo trivijalno rj. $\alpha_1=\alpha_2=\alpha_3=0$

$\Rightarrow 2, \sin x, \cos x \to$ lin. nez.

c) $1 = 1 \cdot \sin^2 x + 1 \cdot \cos^2 x \Rightarrow$ zavisni

$1 - \sin^2 x - \cos^2 x = 0$

$1 \cdot 1 + (-1)\sin^2 x + (-1)\cos^2 x = 0$ – lin. kombinacija je 0, ali nije trivijalna

$\alpha_1 = 1$
$\alpha_2 = -1$
$\alpha_3 = -1$ $\Rightarrow$ lin. zavisni

Furieove fje – nezavisne ANALIZA 3

48. Provjeriti da li su vektori lin. nezavisni: $\sin x, \sin 2x, \sin 3x$? lin. nez.

49. U vektorskom prostoru $\mathbb{R}^4$ je:

$S = L(\{(1,0,-1,2), (-1,2,2,0)\})$ i

$T = L(\{(1,3,-2,0), (2,4,-5,-2)\})$

Odrediti podprostore $S \cap T$ i $S + T$, kao i njihove baze i dimenzije.

W_1, W_2 podprostori p. V

$W_1 \cap W_2 = \{x \in V \mid x \in W_1 \wedge x \in W_2\}$

$W_1 + W_2 = \{x + y \mid x \in W_1 \wedge y \in W_2\}$

Posmatrajmo neki vektor $x \in S \cap T \Rightarrow x \in S \wedge x \in T$

$x \in S \Rightarrow x = \alpha_1 \cdot (1,0,-1,2) + \alpha_2(-1,2,2,0) = (\alpha_1 - \alpha_2, 2\alpha_2, -\alpha_1 + 2\alpha_2, 2\alpha_1)$

$x \in T \Rightarrow x = \beta_1 \cdot (1,3,-2,0) + \beta_2(2,4,-5,-2) = (\beta_1 + 2\beta_2, 3\beta_1 + 4\beta_2, -2\beta_1 - 5\beta_2, -2\beta_2)$

$$\Rightarrow \left.\begin{array}{l} \alpha_1 - \alpha_2 - \beta_1 - 2\beta_2 = 0 \\ 2\alpha_2 - 3\beta_1 - 4\beta_2 = 0 \\ -\alpha_1 + 2\alpha_2 + 2\beta_1 + 5\beta_2 = 0 \\ 2\alpha_1 + 2\beta_2 = 0 \end{array}\right\} \ldots \quad \begin{array}{l} \beta_1 = 2\alpha_1 \\ \beta_2 = -\alpha_1 \\ \alpha_2 = \alpha_1 \end{array}$$

$\Rightarrow x = (0, 2\alpha_1, \alpha_1, 2\alpha_1) = \alpha_1 \cdot (0,2,1,2)$

$\Rightarrow B_{S \cap T} = \{(0,2,1,2)\}$ $\dim S \cap T = 1$

$S+T = L(\{(1,0,-1,2),(-1,2,2,0)(1,3,-2,0)(2,4,-5,-2)\})$

provjeravamo da li su lin. zavisna ova 4 vektora:

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ -1 & 2 & 2 & 0 \\ 1 & 3 & -2 & 0 \\ 2 & 4 & -5 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 2 \\ -1 & 2 & 2 & 0 \\ 1 & 3 & -2 & 0 \\ 3 & 4 & -6 & 0 \end{pmatrix} /\cdot 3 \sim \begin{pmatrix} 1 & 0 & -1 & 2 \\ -1 & 2 & 2 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 10 & 0 & 0 \end{pmatrix} /\cdot(-2)$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 2 \\ -1 & 2 & 2 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

max br. lin. nez. vrsta

rang $A = 3$

$\Rightarrow \dim S+T = 3$

$B = \{(1,0,-1,2),(-1,2,2,0),(0,5,0,0)\}$

T: $\dim(S+T) = \dim S + \dim T - \dim(S\cap T)$

$= 2+2-1 = 3$

68. $\alpha_1 \sin x + \alpha_2 \sin 2x + \alpha_3 \sin 3x = 0$

$x = \frac{\pi}{2}$: $\alpha_1 - \alpha_3 = 0$

$x = \frac{\pi}{4}$: $\frac{\sqrt{2}}{2}\alpha_1 + \alpha_2 + \frac{\sqrt{2}}{2}\alpha_3 = 0$

$x = \frac{\pi}{3}$: $\frac{\sqrt{3}}{2}\alpha_1 + \frac{\sqrt{3}}{2}\alpha_2 = 0$

$$D = \begin{vmatrix} 1 & 0 & -1 \\ \frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \end{vmatrix} = \frac{\sqrt{3}}{2}\cdot\frac{1}{2} \begin{vmatrix} 1 & 0 & -1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & 1 & 0 \end{vmatrix} = \frac{\sqrt{3}}{4}\cdot(0-\sqrt{2}+0+2-\sqrt{2}-0) = \frac{\sqrt{3}}{4}\cdot 2(1-\sqrt{2}) \neq 0$$

sis. ima samo trivijalno rj. $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow$ vektori su lin. nez.

ZADAĆA!

8. Da li je $(\mathbb{R}, \circ)$ grupa?

a) $x \circ y = \sqrt[3]{x^3+y^3} \quad (\forall x, y \in \mathbb{R})$

1° Zatvorenost

$x \circ y \in \mathbb{R}$ za $\forall x, y \in \mathbb{R}$

2° Asocijativnost

$(\forall x, y, z \in \mathbb{R}) \quad (x \circ y) \circ z = x \circ (y \circ z)$ ✓

$\left(\sqrt[3]{x^3+y^3}\right) \circ z = \sqrt[3]{\left(\sqrt[3]{x^3+y^3}\right)^3 + z^3} = \sqrt[3]{x^3+y^3+z^3}$

$x \circ \left(\sqrt[3]{y^3+z^3}\right) = \sqrt[3]{x^3+\left(\sqrt[3]{y^3+z^3}\right)^3} = \sqrt[3]{x^3+y^3+z^3}$

3° Komutativnost ✓

$x \circ y = y \circ x$ za $\forall x, y \in \mathbb{R}$

$x \circ y = \sqrt[3]{x^3+y^3} \qquad y \circ x = \sqrt[3]{y^3+x^3}$

4° Neutralni element ✓

$(\exists e \in \mathbb{R}) (\forall x \in \mathbb{R})$

$x \circ e = e \circ x = x$

$\sqrt[3]{x^3+e^3} = \sqrt[3]{x^3+e^3} \; /^3 \qquad x^3+e^3 = x^3+e^3 \qquad e = 0$

5° Inverzni element ✓

$x \circ x' = x' \circ x = e$

$\sqrt[3]{x^3+x'^3} = \sqrt[3]{x'^3+x^3} = 0 \; /^3$

$x^3 + x'^3 = 0$

$x'^3 = -x^3$

$(x')^3 = (-x)^3$

$x' = -x$

$(\mathbb{R}, \circ)$ je Abelova grupa

b) $x \circ y = \sqrt{x^2 + y^2} \quad (\forall x, y \in \mathbb{R})$

1° Zatvorenost

$\forall x, y \in \mathbb{R} \quad x \circ y \in \mathbb{R}$

2° Asocijativnost ✓

$(x \circ y) \circ z = x \circ (y \circ z)$ $\qquad \sqrt{x^2} = |x|$

$(\sqrt{x^2+y^2}) \circ z = x \circ \sqrt{y^2+z^2}$ $\qquad (\sqrt{x})^2 = x \quad x \in \mathbb{R}$

$\sqrt{(\sqrt{x^2+y^2})^2 + z^2} = \sqrt{x^2 + (\sqrt{y^2+z^2})^2}$

$\sqrt{x^2+y^2+z^2} = \sqrt{x^2+y^2+z^2}$

3° Komutativnost ✓

$x \circ y = y \circ x \qquad \sqrt{x^2+y^2} = \sqrt{y^2+x^2}$

4° Neutralni el. (jedinični)

$x \circ e = e \circ x = x \qquad e = 0 \qquad \sqrt{x^2+e^2} = \sqrt{x^2+0} = \sqrt{x^2} = |x|$

$\sqrt{e^2+x^2} = x!$ za $x<0$ ne vrijedi! a mora vrijediti za $\forall x \in \mathbb{R}$

5° Inverzni el. $\qquad$ pa $(\mathbb{R}, \circ)$ nije grupa!

~~$x \circ x' = x' \circ x = e$~~

~~$\sqrt{x^2+x'^2} = \sqrt{x'^2+x^2} = 0$~~

~~$x^2 + x'^2 = 0 \Rightarrow x^2 = -x'^2 \,/\sqrt{\ }\quad x = ix \notin \mathbb{R}$~~

~~$(\mathbb{R}, \circ)$ nije grupa, $(\mathbb{R}, \circ)$ je polugrupa sa jedinicom~~

11. $(a,b)\circ(c,d)=(a+c+1,\ b+d)$

$(a,b)*(c,d)=(ac-bd+a+c,\ bc+ad+b+d)$

$(\mathbb{R}^2,\circ,*)$ komutativni prsten sa jedinicom

a) $(\mathbb{R}^2,\circ)$ Abelova grupa ✓

b) $(\mathbb{R},*)$ polugrupa (kom. polugrupa sa 1 ?✓) ✓

c) lijevi i desni zakon distributivnosti $*$ prema $\circ$ ✓

a) 1° zatvorenost

$\forall (a,b),(c,d)\in\mathbb{R}^2 \quad (a,b)\circ(c,d)\in\mathbb{R}^2$

2° asocijativnost ✓

$\forall (a,b),(c,d),(e,f)\in\mathbb{R}^2$

$((a,b)\circ(c,d))\circ(e,f)=(a,b)\circ((c,d)\circ(e,f))$

$(a+c+1,\ b+d)\circ(e,f)=(a+c+1+e+1,\ b+d+f)=(a+c+e+2,\ b+d+f)$

$(a,b)\circ(c+e+1,\ d+f)=(a+c+e+1+1,\ b+d+f)=(a+c+e+2,\ b+d+f)$

3° komutativnost ✓

$\forall (a,b),(c,d)\in\mathbb{R}^2 \qquad (a,b)\circ(c,d)=(c,d)\circ(a,b)$

$(a+c+1,\ b+d)=(c+a+1,\ b+d)$

4° neutralni el.

$(a,b)\circ(e_1,e_2)=(a,b)$

$(a+e_1+1,\ b+e_2)=(a,b)$

$\cancel{a}+e_1+1=\cancel{a} \quad \wedge \quad \cancel{b}+e_2=\cancel{b}$

$e_1=-1 \qquad e_2=0 \qquad (e_1,e_2)=(-1,0)$

5° inverzni el.

$(a,b)\circ(a',b')=(e_1,e_2) \qquad (a,b)'=(-2-a,\ -b)$

$(a+a'+1,\ b+b')=(-1,0)$

$a+a'+1=-1 \quad \wedge \quad b+b'=0$

$a=-a'-2 \quad a'=-2-a \qquad b'=-b \quad \boxed{b=-b'}$

b) 1° zatvorenost

$\forall (a,b),(c,d) \in \mathbb{R}^2 \quad (a,b)*(c,d) \in \mathbb{R}^2$

2° asocijativnost

$\forall (a,b),(c,d),(e,f) \in \mathbb{R}^2 \quad ((a,b)*(c,d))*(e,f) = (a,b)*((c,d)*(e,f))$

$((a,b)*(c,d))*(e,f) = (ac-bd+a+c,\ bc+ad+b+d)*(e,f) =$

$= (e(ac-bd+a+c) - f(bc+ad+b+d) + ac-bd+a+c+e,$

$e(bc+ad+b+d) + f(ac-bd+a+c) + bc+ad+b+d+f)$

$(a,b)*((c,d)*(e,f)) = (a,b)*(ce-df+c+e,\ de+cf+d+f) =$

$= (a(ce-df+c+e) - b(de+cf+d+f) + a + ce-df+c+e,$

$b(ce-df+c+e) + a(de+cf+d+f) + b + de+cf+d+f)$

$(ace - bde + ae + ce - fbc - fad - fb - fd + ac - bd + a + c + e,$ ✓

$ebc + ead + eb + ed + fac - fbd + fa + fc + bc + ad + b + d + f)$ ✓

3° komutativnost ✓

$\forall (a,b),(c,d) \in \mathbb{R}^2 \quad (a,b)*(c,d) = (c,d)*(a,b)$

$(ac-bd+a+c,\ bc+ad+b+d) = (ca-db+c+a,\ ad+cb+d+b)$

4° neutralni el.

$\forall (a,b) \in \mathbb{R}^2$

$(a,b)*(e_1,e_2) = (e_1,e_2)*(a,b) = (a,b)$

$(ae_1 - be_2 + a + e_1,\ be_1 + ae_2 + b + e_2) = (a,b)$

$ae_1 - be_2 + \cancel{a} + e_1 = \cancel{a} \quad \wedge \quad be_1 + ae_2 + \cancel{b} + e_2 = \cancel{b}$

$e_1(a+1) = be_2 \qquad e_2(a+1) = -be_1$

$e_1 = \dfrac{be_2}{a+1} \qquad e_2(a+1) + b \cdot \dfrac{be_2}{a+1} = 0$

$e_2(a+1)^2 + b^2 e_2 = 0$

$e_2(b^2 + (a+1)^2) = 0$

$e_2 = 0 \Rightarrow e_1 = 0$

$\Rightarrow (e_1,e_2) = (0,0)$

c) $(a,b) * [(c,d) \circ (e,f)] = [(a,b)*(c,d)] \circ [(a,b)*(e,f)]$

$(a,b) * (c+e+1, d+f) = [ac-bd+a+c, bc+ad+b+d] \circ [ae-bf+a+e, be+af+b+f]$

$(ac+ae+a-bd-bf+a+c+e+1, bc+be+b+ad+af+b+d+f) =$

$= ac-bd+a+c+ae-bf+a+e+1, bc+ad+b+d+be+af+b+f)$

$\Rightarrow (\mathbb{R}^2, \circ, *)$ je komutativni prsten sa jedinicom

LINEARNI OPERATORI

$f: V_1 \to V_2$

$(\forall \alpha,\beta \in F)(\forall x,y \in V_1)\ f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

50. Neka je M_2 vekt. p. kvadratnih matrica nad F, i $f: M_2 \to M_2$ definisano sa: $(\forall X \in M_2)\ f(X) = X \cdot M$ pri čemu je $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ fiksirana matrica iz M_2

a) Dokazati da je f linearni operator.

b) Odrediti matricu operatora F u odnosu na bazu: $\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_4} \right\}$

a) $X, Y \in M_2,\ \alpha, \beta \in F$

$f(\alpha X + \beta Y) = (\alpha X + \beta Y) \cdot M = (\alpha X)M + (\beta Y)M$

$= \alpha(XM) + \beta(YM) = \alpha f(X) + \beta f(Y)$

$V_1;\ B_1 = \{e_1, \dots, e_m\};\quad \dim V_1 = m$

$V_2;\ B_2 = \{f_1, \dots, f_n\};\quad \dim V_2 = n$

$(\forall x \in V_1)\quad f(x) = Ax$